Journal of Engineering Mathematics **46:** 241–252, 2003.

Spatial-decay estimates for a generalized biharmonic equation in inhomogeneous elasticity

J. N. FLAVIN
Department of Mathematical Physics, National University of Ireland, Galway, Ireland
(e-mail: james.flavin@nuigaway.ie)

Received 8 April 2002; accepted in revised form 14 March 2003

Abstract. A rectangular region of smoothly varying, inhomogeneous, isotropic elastic material is considered; two types of material are dealt with: if opposite pairs of edges are parallel to the x, y axes, in one case the elastic moduli vary smoothly with x, while in the other they vary smoothly with y. The region is in a state of plane strain, three of its edges being traction-free, the fourth being subjected to a self-equilibrated, in-plane, load. Inequality estimates are obtained descriptive of the spatial decay of effects away from the loaded end. The results of the paper imply how the estimated decay rate varies with the constitutive profile, and may have applications to functionally graded materials.

Key words: inhomogeneous istropic elastic material, plane strain, spatial-decay estimates, Saint Venant's principle.

1. Introduction

One of the areas where the biharmonic equation has proved most useful is that of two-dimensional elasticity: essentially, the determination of the stress components in the context of plane strain and of generalized plane stress (introduced by Filon) for a homogeneous isotropic elastic material is reducible to the solution of this equation under suitable boundary conditions (in the case of a simply connected region). The corresponding issues for (smoothly varying) *inhomogeneous* isotropic elastic material lead to a generalized biharmonic equation. Such issues – which have applications to the technologically important functionally graded materials – form the subject of this paper.

This paper considers a rectangular strip consisting of smoothly varying inhomogeneous, isotropic elastic material in an equilibrium state of plane strain, three of its edges being traction-free and the remaining one – corresponding to $x = 0$ – being (necessarily) subjected to a self-equilibrated (in plane) load. The objective is to derive inequality estimates reflecting decay of effects away from the loaded end. Two types of (suitably restricted) smoothly varying inhomogeneity are considered:

(i) the elastic moduli vary smoothly with the rectangular coordinate y,

(ii) they vary smoothly with the rectangular coordinate x,

the x, y coordinates being in the directions of the edges of the rectangular region.

In the case of the inhomogeneity of type (i) the inequality estimate derived is of the cross-sectional type based on a second-order differential inequality. An inequality estimate of the 'energy' type based on a differential-integral is also quoted, this being a slight generalization of that dealt with in [1]. The estimated decay rate arising from the two approaches is the same. In the case of the inhomogeneity of type (ii) the inequality estimate derived is of the 'energy' type based on a differential-integral inequality.

Spatial-decay estimates in two-dimensional elasticity date back to the seminal paper of Knowles [2]. Comprehensive reviews of these and other spatial-decay estimates in elasticity, and in the more general context of elliptic equations, may be found in [3–6]. The general methodologies used in this paper are due to Knowles [2], and to Flavin and Knops [7]; the former deals with an inequality estimate for an energy-like measure, based on a differential-integral inequality, while the latter deals with an inequality estimate for a positive-definite cross-sectional measure, based on a second-order differential inequality estimate.

It should be noted that the smoothly varying inhomogeneous elastic materials considered here provide a model for technologically important FGMs- functionally graded materials. These materials have received considerable attention in recent literature, *e.g.* [8–12]. Spatial-decay estimates in the context of anti-plane elastic shear deformations have been studied intensively in [10–12].

Whereas the paper discusses plane strain only all the analyses given are easily modified to cater for the case of generalized plane stress.

2. Notation and equations

The Airy stress function $\phi(x, y)$ is introduced to simplify the analysis, and is such that the (relevant) stress components $\tau_{xx}, \tau_{xy}, \tau_{yy}$ are given by

$$\tau_{xx} = \phi_{yy},\ \tau_{yy} = \phi_{xx},\ \tau_{xy} = -\phi_{xy}, \tag{1}$$

where subscripts attached to ϕ mean partial differentiation with respect to the appropriate variables, both here and subsequently. It proves convenient – in the case of both inhomogeneities considered – to work with elastic moduli $\varepsilon, \bar{\varepsilon}$ related to Young's modulus E, Poisson's ratio σ by means of

$$\varepsilon = (1 - \sigma^2)E^{-1}, \quad \bar{\varepsilon} = \sigma(1 - \sigma)^{-1}\varepsilon\ . \tag{2}$$

It is assumed throughout that

$$E > 0 \quad 0 \leq \sigma \leq 1/2 \tag{3}$$

and that thereby

$$\varepsilon > 0, \quad \bar{\varepsilon} \geq 0\,. \tag{4}$$

It should be noted that in the case of incompressibility $(\sigma \to 1/2)$ $\bar{\varepsilon} \to \varepsilon$.

Throughout the paper a rectangular region $0 < x < L$, $0 < y < 1$ is considered, (x, y) denoting rectangular Cartesian coordinates. It is supposed to be occupied by smoothly varying inhomogeneous isotropic elastic material in an equilibrium state of plane strain, the edges $x = L$, $y = 0, 1$, being traction-free, the remaining edge being (necessarily) subject to a self-equilibrated load. Sometimes a semi-infinite rectangular strip is envisaged $(L \to \infty)$. We adopt the following notation: R_x denotes the sub-rectangle between abscissae (x, L), R_0 denoting the entire rectangle, and L_x denotes the line within the rectangular region with abscissa x, parallel to and in the same sense as the y-axis.

Ordinary differentiations with respect to x and y are denoted throughout by superposed primes and dots, respectively. In the case of inhomogeneities of type (i) (for which $\varepsilon, \bar{\varepsilon}$ are smooth functions of y) the equation satisfied by ϕ is easily shown to be

$$(\varepsilon\phi_{xx})_{xx} + 2(\varepsilon\phi_{xy})_{xy} + (\varepsilon\phi_{yy})_{yy} - \ddot{\bar{\varepsilon}}\,\phi_{xx} = 0, \tag{5}$$

while in the case of inhomogeneities of type (ii) (for which $\varepsilon, \bar{\varepsilon}$ are smooth functions of x) it is

$$(\varepsilon\phi_{xx})_{xx} + 2(\varepsilon\phi_{xy})_{xy} + (\varepsilon\phi_{yy})_{yy} - \bar{\varepsilon}''\phi_{yy} = 0. \tag{6}$$

These hold in the rectangular region R_0, while on its boundary the arbitrariness inherent in ϕ may be used to give the simplified boundary conditions

$$\left.\begin{aligned} &\phi = \phi_y = 0 \text{ on the edges } y = 0, 1, \\ &\phi = \phi_x = 0 \text{ on the edge } x = L\ . \end{aligned}\right\} \tag{7}$$

It is assumed throughout that ϕ is a smooth function [*e.g.* $\phi \in C^4(\overline{R}_0)$].

3. Inhomogenous material of type 1

In this section we consider inhomogeneous material of the type (i): $\varepsilon = \varepsilon(y), \bar{\varepsilon} = \bar{\varepsilon}(y)$, where both of these are smooth functions. We define the cross-sectional measure of stress as follows:

$$F(x) = \int_0^1 \varepsilon(\phi_{xx}^2 + \phi_{yy}^2)\,\mathrm{d}y. \tag{8}$$

Straightforward differentiation, use of (5), (7) together with integration by parts gives

$$F''(x) = 2\int_0^1 \varepsilon[\phi_{xxx}^2 + \phi_{yyx}^2 + 2\phi_{xxy}^2]\mathrm{d}y + 2\int_0^1 \ddot{\bar{\varepsilon}}\,\phi_{xx}^2\,\mathrm{d}y\,, \tag{9}$$

primes denoting (ordinary) differentiation with respect to x, superposed dots differentiation with respect to y. Let us *assume* henceforward that

$$\ddot{\bar{\varepsilon}} \geq 0,$$

i.e., that $\bar{\varepsilon}$ is convex. In these circumstances it is clear that F is convex in x. Using (8), (9) together with inequality (65), we obtain

$$F''(x) \geq 2\int_0^1 \varepsilon[\phi_{xxx}^2 + \phi_{yyx}^2 + 2\lambda_1\phi_{xx}^2]\,\mathrm{d}y, \tag{10}$$

where λ_1, here and subsequently, denotes the lowest positive eigenvalue of

$$\ddot{u} + [\lambda - (\varepsilon^{1/2})^{\cdot\cdot}\varepsilon^{-1/2}]\,u = 0\,,\quad y \in (0,1)\,;\quad u(0) = u(1) = 0. \tag{11}$$

The quantity λ_1 will subsequently be seen to determine the (estimated) decay rate. Whereas one cannot, in general, obtain λ_1 explicity, let us note, *en passant*, the easily derived, crude bound

$$\lambda_1 \geq \pi^2\varepsilon_{\min}/\varepsilon_{\max},$$

the quantities on the right-hand side denoting minimum and maximum values; this is valid even when $\ddot{\varepsilon}$ fails to exist. What is central to this paper is the manner in which λ_1 varies with the constitutive profile (see Remarks 2,3). The aforementioned property is not the most compelling one describing the variation of λ_1 with the constitutive profil; these are implicit in Remarks 2,3 and in Section 5.

With a view to establishing that $F(x)$ satisfies a generalised convexity condition, one requires the following conservation law:

$$\int_0^1 [\varepsilon\left\{2\phi_x\phi_{xxx} - \phi_{xx}^2 + \phi_{yy}^2 - 2\phi_{xy}^2\right\} - \ddot{\varepsilon}\,\phi_x^2]\,\mathrm{d}y = E, \tag{12}$$

where E is a constant; this may be derived in an elementary manner by multiplying (5) by ϕ_x and integrating with respect to y. Application of (12) to the unloaded end $x = L$ readily establishes that $E \leq 0$. Using this together with (12) and the arithmetic -geometric inequality, we recover the inequality

$$\int_0^1 \varepsilon\phi_{xx}^2\,\mathrm{d}y \ \geq \int_0^1 [\varepsilon\{\phi_{yy}^2 - 2\phi_{xy}^2 - \theta\phi_x^2 - \theta^{-1}\phi_{xxx}^2\} - \ddot{\varepsilon}\,\phi_x^2]\,\mathrm{d}y\,, \tag{13}$$

where θ is any positive constant. This, together with (10), readily yields

$$\begin{aligned} F''(x) \geq\ & (2\lambda_1 - \delta)\int_0^1 \varepsilon\phi_{xx}^2\,\mathrm{d}y + \delta\int_0^1 \varepsilon\phi_{yy}^2\mathrm{d}y \\ & + \int_0^1 [\varepsilon\{2\phi_{xyy}^2 + (2 - \delta\theta^{-1})\phi_{xxx}^2 - 2\delta\phi_{xy}^2 - \delta\theta\phi_x^2\} - \delta\,\ddot{\varepsilon}\,\phi_x^2]\,\mathrm{d}y, \end{aligned} \tag{14}$$

where δ is any positive constant. Choosing $\theta = \delta/2$, we obtain

$$\begin{aligned} F''(x) \geq\ & (2\lambda_1 - \delta)\int_0^1 \varepsilon\phi_{xx}^2\,\mathrm{d}y + \delta\int_0^1 \varepsilon\phi_{yy}^2\mathrm{d}y \\ & + \int_0^1 [\varepsilon\{2\phi_{xyy}^2 - 2\delta\phi_{xy}^2 - \frac{1}{2}\delta^2\phi_x^2]\,\mathrm{d}y - \delta\int_0^1 \ddot{\varepsilon}\,\phi_x^2\,\mathrm{d}y. \end{aligned} \tag{15}$$

The last term in (15) is slightly troublesome. Whereas it can be handled in a number of ways, we choose the way that appears to give the most transparent result (In this connection, see Remark 2). Integration by parts etc., use of Schwarz's inequality, use of inequality (65) yields

$$-\int \ddot{\varepsilon}\,\phi_x^2\,\mathrm{d}y \ \geq -4\ \Sigma\lambda_1^{-1}\int_0^1 \varepsilon\phi_{xyy}^2\,\mathrm{d}y\,, \tag{16}$$

where

$$\Sigma = \sigma_M(1 - \sigma_M)^{-1}, \tag{17}$$

and where σ_M denotes the maximum value of σ arising. Simpler, but cruder, estimates follow on taking the permissible value $\Sigma = 1$ (*cf.* [1]).

It follows from (15), (16) and further use of inequality (65) that

$$\begin{aligned} F''(x) \geq\ & (2\lambda_1 - \delta)\int_0^1 \varepsilon\phi_{xx}^2\mathrm{d}y + \delta\int_0^1 \varepsilon\phi_{yy}^2\,\mathrm{d}y \\ & + (1/2)\,[4 - 4(1 + 2\Sigma)(\delta\lambda_1^{-1}) - (\delta\lambda_1^{-1})^2]\ \int_0^1 \varepsilon\phi_{xyy}^2\,\mathrm{d}y. \end{aligned} \tag{18}$$

Choosing δ to be the largest root of the quadratic term in (18), we have

$$\delta = r\lambda_1, \tag{19}$$

where

$$r = 2\left\{\sqrt{(1+2\Sigma)^2+1} - (1+2\Sigma)\right\} . \tag{20}$$

It follows from (8), (18–20) that

$$F''(x) - r\lambda_1 F(x) \geq 0\,, \tag{21}$$

where r is given by (20).

It follows from this and a well-known *Comparison Principle* (a generalisation of the curve under chord property for convex functions; *e.g.* [13, p. 124]) that $F(x)$ is bounded above by $G(x)$, the solution of the differential *equation* corresponding to the differential inequality (21) and the same boundary conditions. This yields

$$\begin{aligned} F(x) &\leq [F(0)\sinh\{\sqrt{r\lambda_1}(L-x)\} + F(L)\,\sinh\{\sqrt{r\lambda_1}x\}]\,/\,\sinh(\sqrt{r\lambda_1}L) \\ &\leq F(0)\mathrm{e}^{-\sqrt{r\lambda_1}\,x} + F(L)\mathrm{e}^{-\sqrt{r\lambda_1}(L-x)}, \end{aligned} \tag{22}$$

where the latter step follows by elementary analysis.

This result is not entirely satisfactory as

$$F(L) = \int_0^1 \tau_{yy}^2(L, y)\,\mathrm{d}y$$

is not generally known. We let $L \to \infty$ and obtain a decay result (reminiscent of a Phragmén-Lindelöf result) which we embody in a Theorem:

Theorem 1. *In the context of a rectangular strip* $0 < x < L,\ 0 < y < 1$, *for which* $L \to \infty$, *consisting of inhomogeneous material of* Type 1 *for which* $\bar{\varepsilon}$ *is convex, the cross-sectional stress measure* (8) *satisfies the decay law*

$$F(x) \leq F(0)\mathrm{e}^{-\sqrt{r\lambda_1}\,x} \tag{23}$$

where r *is defined by* (20), (17) *and* λ_1 by (11), *provided that*

$$\lim_{L\to\infty} \mathrm{e}^{-\sqrt{r\lambda_1}\,L} \int_0^1 \tau_{yy}^2(L, y)\,\mathrm{d}y = 0\,. \tag{24}$$

Remark 1. It will be noted that (24) is satisfied in particular if

$$\phi_{xx} \to 0 \ \ as \quad x \to \infty. \tag{25}$$

Remark 2. Note that the decay rate given by (23) coincides with that of Knowles [2] in the case of homogeneous materials with vanishing Poisson's ratio. Whereas it is possible to do better than this in *particular* contexts *our aim is to get a relatively simple result which will give an overall view of how the estimated decay rate depends on varying constitutive profiles e.g.* [1].

Remark 3. The estimated decay rate is essentially the eigenvalue λ_1 of (11) (see also the

Appendix). The dependence of λ_1 on the constitutive profile is discussed in [1], and, in the context of the analogous anti-plane shear problem, in [10–12]. In the present context, for example, λ_1 is, in a sense, a monotonically increasing function of $(\varepsilon^{1/2})^{\cdot\cdot}\varepsilon^{-1/2}$.
Remark 4. The estimate (23) can be made explicit in terms of conventional data in either of the following two circumstances [*cf.* [7]]:
(a) There is normal loading (*i.e.,* $\tau_{xy} = 0$)on the edge $x = 0$ and the (additional) asymptotic condition (25) obtains. One then has (via the *Conservation Law* (12))

$$F(0) = 2\int_0^1 \tau_{xx}^2\,(0, y)\,\mathrm{d}y\,. \tag{26}$$

(b) The normal-stress component and the complementary, tangential displacement component are specified on the edge $x = 0$.
Remark 5. The energy approach has been used in connection with the above problem in [1]. In order to compare the results arising from the two approaches, we quote a very easily generalised version of the analysis occurring in [1]. Let R_x denote the portion of the rectangle $0 < x < L,\ 0 < y < 1$ betwen abscissae x, L. Define the (positive definite) global measure of stress in R_x :

$$E(x) = \int_{R_x} [\varepsilon\phi_{xx}^2 + 2\varepsilon\phi_{xy}^2 + \varepsilon\phi_{yy}^2 + \ddot{\bar{\varepsilon}}\ \phi_x^2]\,\mathrm{d}y\,, \tag{27}$$

where one again *assumes* $\ddot{\bar{\varepsilon}} \geq 0$ (in order that $E(x)$ be a positive definite measure). One has

$$E(x) \leq 2E(0)\ \exp[-\sqrt{r\lambda_1}x] \tag{28}$$

where r,λ_1 are as previously defined, or [*cf.* [14]) in the case of a semi-infinite region ($L \to \infty$)

$$E(x) \leq E(0)\exp[-\sqrt{r\lambda_1}\,x]\,. \tag{29}$$

It is apparent that the decay rate arising here coincides with that arising in Theorem 1.
Moreover, estimates (28), (29) may be made fully explicit on noting that

$$E(0) \leq \int_{R_x} [\varepsilon\ \bar{\phi}_{xx}^2\ + 2\varepsilon\ \bar{\phi}_{xy}^2\ + \varepsilon\ \bar{\phi}_{yy}^2\ + \ddot{\bar{\varepsilon}}\bar{\phi}_x^2]\mathrm{d}y,$$

where $\bar{\phi}$ is any smooth function taking the same boundary conditions as ϕ see [1]
The restriction $\ddot{\bar{\varepsilon}} \geq 0$ can be relaxed somewhat [1] – both in the current context and in the general context of this section, but at the price of a considerably more complicated analysis.
Moreover, this estimate is easily shown to be applicable *mutatis mutandis* to a fairly general simply connected region.

4. Inhomogeneous material of type 2

The material is supposed to be inhomogeneous of type (ii) in this section: it is supposed that $\varepsilon, \bar{\varepsilon}$ are both smooth functions of x. We proceed to obtain a decay estimate of the 'energy' type.
Multiplying the p.d.e. (5) by ϕ and integrating over R_x [the region between the abscissae x and L], gives rise to the following: The quantity defined by

$$E(x) = \int_{R_x} [\varepsilon(\phi_{xx}^2 + \phi_{yy}^2 + 2\phi_{xy}^2) + \bar{\varepsilon}'' \ \phi_y^2]\,\mathrm{d}A \tag{30}$$

may also be expressed as

$$E(x) = -\int_{L_x} [\varepsilon\phi_x\phi_{xx} - \phi(\varepsilon\phi_{xx})_x + 2\varepsilon\phi_y\phi_{xy}]\mathrm{d}y. \tag{31}$$

Plainly $E(x)$, defined by (30), is positive-definite in ϕ provided

$$\bar{\varepsilon}'' \geq 0,$$

(*i.e.*, $\bar{\varepsilon}$ is a convex function of x), or, more generally, provided

$$\bar{\varepsilon}'' > -4\pi^2\varepsilon \tag{32}$$

in view of inequality (66) of the Appendix. The constitutive restriction (32) is *assumed* – as a minimum – henceforward, and, in these circumstances, the positive definiteness referred to, qualifies (30) as a suitable global measure of stress in R_x.

Proceeding as in Knowles [2], we find that

$$E^{'}(x) = -\int_{L_x} [\varepsilon(\phi_{xx}^2 + \phi_{yy}^2 + 2\phi_{xy}^2) + \bar{\varepsilon}''\phi_y^2)]\,\mathrm{d}A \tag{33}$$

and, after a little manipulation, that

$$\int_x^L E(x)\mathrm{d}x = \int_{L_x} \varepsilon(\phi_x^2 + \phi_y^2 - \phi\phi_{xx})\,\mathrm{d}y + \int_{R_x} \varepsilon'(\phi_x^2 + \phi_y^2)\,\mathrm{d}A. \tag{34}$$

Let us *assume pro tem.* that the additional constitutive restriction

$$\varepsilon' \leq 0, \tag{35}$$

holds, thereby obtaining

$$\int_x^L E(x)\,\mathrm{d}x \leq \int_{L_x} \varepsilon(\phi_x^2 + \phi_y^2 - \phi\phi_{xx})\,\mathrm{d}y. \tag{36}$$

If K is any (positive) constant, we obtain from (33), (36), on completing a square, that

$$\begin{aligned} E'(x) + 4K^2 \textstyle\int_x^L E(x)\,\mathrm{d}x \leq -\int_{L_x} [\,&\varepsilon(2\phi_{xy}^2 - 4K^2\phi_x^2) \\ &+\varepsilon(x)\{\phi_{yy}^2 - (4K^2 - \bar{\varepsilon}''\,\varepsilon^{-1})\phi_y^2 - 4K^4\phi^2\}]\,\mathrm{d}y. \end{aligned} \tag{37}$$

Applying Inequalities (65), (66) we deduce

$$\begin{aligned} E'(x) + 4K^2 \int_x^L E(x)\,\mathrm{d}x &\leq -\varepsilon(x) \int_{L_x} (2\pi^2 - 4K^2)\,\phi_x^2\,\mathrm{d}y \\ &- \int_{L_x} \{4\pi^2 - (4K^2 - \bar{\varepsilon}''\,\varepsilon^{-1}) - 4\pi^{-2}K^4\}\phi_y^2\ \mathrm{d}y. \end{aligned} \tag{38}$$

We may recover the same decay law as one obtains for homogeneous material (*e.g.* [15]) if the (more restrictive) constitutive restriction

$$\pi^2 + \bar{\varepsilon}'' \varepsilon^{-1} \geq 0 \tag{39}$$

applies (in place of (32)). If one chooses

$$K = \pi/\sqrt{2}, \tag{40}$$

the first term on the right-hand side of (38) vanishes, and the second term is non-positive provided that (39). On writing

$$F(x) = E(x) + 2K \int_x^L E(x)\,\mathrm{d}x \tag{41}$$

we obtain

$$F' + 2KF \leq 0, \tag{42}$$

and recover, as in [2], [15],

$$E(x) \leq 2E(0)(1 + e^{-4KL})^{-1} \exp(-\pi/\sqrt{2}). \tag{43}$$

An alternative result can similarly be obtained when (39) is replaced by a different (complementary) constitutive restriction. One readily verifies that the right-hand side of (38) is non-positive if

$$K = \frac{\pi}{\sqrt{2}}[\sqrt{5 + (\bar{\varepsilon}''\ \varepsilon^{-1})_m \pi^{-2}} - 1], \tag{44}$$

where $(\bar{\varepsilon}''\varepsilon^{-1})_m$ is the minimum of $\bar{\varepsilon}''\varepsilon^{-1}$ arising, provided that

$$-\pi^2 > \bar{\varepsilon}''\varepsilon^{-1} > -4\pi^2. \tag{45}$$

One again has

$$E(x) \leq 2E(0)\,\{1 + \mathrm{e}^{-4KL}\}^{-1} \exp(-2Kx), \tag{46}$$

where K is now given by (44).

The foregoing results may be embodied in a Theorem:

Theorem 2. *In the context of inhomogeneous material of Type* 2 *for which* $\varepsilon(x)$ *satisfies*

$$\varepsilon'(x) \leq 0, \tag{47}$$

one has

$$E(x) \leq 2E(0)\{1 + \mathrm{e}^{-4KL}\}^{-1} \exp(-2Kx) \tag{48}$$

$$\leq 2E(0) \exp(-2Kx), \tag{49}$$

where

(i)

$$K = \pi/\sqrt{2}, \tag{50}$$

when the additional constitutive restriction

$$\pi^2 + \bar{\varepsilon}''\varepsilon^{-1} \geq 0 \tag{51}$$

holds;
(ii) *where*

$$K = \frac{\pi}{\sqrt{2}}\,[\sqrt{5 + (\overline{\varepsilon}''\varepsilon^{-1})_m \pi^{-2}}\; - 1\,], \tag{52}$$

where $(\overline{\varepsilon}''\varepsilon^{-1})_m$ *means the minimum value of* $\overline{\varepsilon}''\varepsilon^{-1}$ *arising, when the additional constitutive restriction*

$$-\pi^2 > \overline{\varepsilon}''\varepsilon^{-1} > \; -4\pi^2 \tag{53}$$

holds.
Remark 6. In the case of a semi-infinite rectangular strip ($L \to \infty$) one may prove that one may remove the premultiplying factor of 2 occurring in (49) (*e.g.* [14], Theorem 2 following).

We now prove a result appropriate to a different set of constitutive restrictions, dropping (35) in particular, but retaining (51). Denoting by $\left|(\varepsilon'\varepsilon^{-1})_+\right|_m$ the maximum positive value of $\varepsilon'\varepsilon^{-1}$ arising, we may prove – proceeding as previously – that

$$E'(x) + 4K^2 \int_x^L E(x)\,\mathrm{d}x \; - 2K^2\pi^{-2}\left|(\varepsilon'\varepsilon^{-1})_+\right|_m E(x) \;\; \leq 0, \tag{54}$$

where $K = \pi/\sqrt{2}$. The integro-differential inequality (54) (for $L \to \infty$) of the type

$$E'(x) + 4K^2 \int_x^\infty E(x)\,\mathrm{d}x \; - 4K^2\gamma\, E(x) \leq 0, \tag{55}$$

γ being a constant, considered by Vafeades and Horgan [14] in the context of a semi-infinite von Karman plate. When a semi-infinite region is considered in the current context, the analysis of the aforementioned paper continues to be valid, giving the following theorem:

Theorem 3. *In the context of a semi-infinite retangular region* ($L \to \infty$) *consisting of inhomogeneous material of type* 2*, for which*

$$\bar{\varepsilon}''\varepsilon^{-1} + \pi \geq 0, \tag{56}$$

one has

$$E(x) \leq E(0)\mathrm{e}^{-mx}, \tag{57}$$

with

$$m = \pi\sqrt{2}(\sqrt{1 + M^2} - M), \tag{58}$$

where

$$M = \left|(\varepsilon'\varepsilon^{-1})_+\right|_m (2\sqrt{2}\pi)^{-1}, \tag{59}$$

$\left|(\varepsilon'\varepsilon^{-1})_+\right|_m$ *denoting the maximum positive value of* $\varepsilon'\varepsilon^{-1}$ *arising.*

Remark 7. The foregoing estimate (57) degenerates for large values of $\left|(\varepsilon'\varepsilon^{-1})+\right|_m$. It should be noted also that variants of the estimates (57), arise if the constitutive restriction (56) is altered, *e.g.*, if (56) is replaced by the convexity restriction $\bar{\varepsilon}'' \geq 0$, (57) continues to hold but with a value of M half that given by (59).

We now discuss how the previous estimates for $E(x)$ may be made fully explicit by showing how to bound $E(0)$ above in terms of data. Let $\overline{\phi}$,$\bar{\psi}$ be any two smooth functions having the same boundary values as ϕ. Consider the scalar product (R_0 being the entire region)

$$(\bar{\phi},\ \bar{\psi}) = \int_{R_O} [\varepsilon\overline{\phi}_{xx}\overline{\psi}_{xx} + \varepsilon\overline{\phi}_{yy}\overline{\psi}_{yy} + 2\varepsilon\overline{\phi}_{xy}\overline{\psi}_{xy} + \overline{\varepsilon}''\ \overline{\phi}_y\,\overline{\psi}_y]\,\mathrm{d}A \tag{60}$$

and suppose

$$\overline{\varepsilon}''\varepsilon^{-1} > -4\pi^2\ . \tag{61}$$

It is easily verified, under the latter restriction (bearing in mind (66)), that

$$(\overline{\phi}, \overline{\phi})\ \geq 0 \tag{62}$$

with equality iff $\overline{\phi} = 0$. One easily verifies by integration by parts that

$$(\overline{\phi}, \phi)\ = (\phi, \phi). \tag{63}$$

In view of the properties of the scalar product and (62), Schwarz's inequality (as applied to $\overline{\phi}, \phi$) may be used to prove that

$$(\phi, \phi) \leq (\overline{\phi}, \overline{\phi})\ \ . \tag{64}$$

This result is embodied in an auxiliary Theorem:
Theorem 4. *The inequality estimates* (48), (49) and (57) *may be made fully explicit in terms of data prescribed on the edge* $x = 0$ [ϕ, ϕ_x *being available thereon in terms of the specified self-equilibrated load*] *as follows*:

$$E(0) \leq\ (\overline{\phi}, \overline{\phi})\,,$$

where $\overline{\phi}$ *is a smooth function satisfying the same boundary conditions as* ϕ *and where the scalar product* $(.,.)$ *is defined by* (60), *provided that the constitutive restriction* (61) holds.

5. Conclusions

5.1. (A) MATERIAL OF TYPE 1

In connection with the decay estimates for the material of type 1 (smooth variation of elastic moduli with y), the estimated decay rate essentially depends (in certain circumstances) on the eigenvalue λ_1, defined by [11]. The results of quoted papers show how this varies with the constitutive profile of the material, *e.g.* the following are available in [1]:

(i) The quantity $\lambda_1(\varepsilon)$ is monotonically increasing in the sense: if $(\varepsilon_1^{1/2})^{\cdot\cdot}\varepsilon_1^{-1/2} > (\varepsilon_2^{1/2})^{\cdot\cdot}\varepsilon_2^{-1/2}$, the corresponding eigenvalues λ_1, λ_2 (say) satisfy $\lambda_1 > \lambda_2$.

(ii) If $(\varepsilon^{1/2})^{\cdot\cdot} = 0$, then $\lambda_1 = \pi^2$. If $(\varepsilon^{1/2})^{\cdot\cdot}\varepsilon^{-1/2} \geq k$ (const.) $\geq 0, \lambda_1 \geq \pi^2 + k$; if $(\varepsilon^{1/2})^{\cdot\cdot}\varepsilon^{-1/2} \leq -k$ (const.) $\leq 0, \lambda_1 \leq \pi^2 - k$; the equality signs coincide.

(iii) The content of (i) may be expressed in an intuitive manner: suppose $(\varepsilon^{1/2})^{\cdot\cdot} \geq 0$, then the more 'convex' $\varepsilon^{1/2}(y)$ is – towards the y-axis – the greater is λ_1; suppose $(\varepsilon^{1/2})^{\cdot\cdot} \leq 0$, then the more 'concave' $\varepsilon^{1/2}(y)$ is – towards the y-axis – the smaller is λ_1.

(iv) Suppose, for definitesness, that the materials properties are symmetric with respect to $y = 1/2$, then $(\varepsilon^{1/2})^{\cdot\cdot}\varepsilon^{-1/2}$ positive and 'large', implies 'large' λ_1, and corresponds to relatively 'hard' material in the centre with realtively 'soft' layers outside. In interpreting 'hardness' and 'softness', we mean 'small' and 'large' values respectively of $(1 - \sigma^2)E^{-1}$.

5.2. (B) MATERIAL OF TYPE 2

In connection with the decay estimates for the material of type 2 (elastic moduli varying smoothly with respect to x), the dependence of the estimated decay constant on the constitutive profile of the material is complicated, and not easily verbalized (at least in the generality addressed in the paper). However, the implications of Theorem 2 in a limited – but perhaps not unrepresentative – set of circumstances are addressed: consider an incompressible material, or one with constant Poisson's ratio, that 'hardens' with x, essentially characterized by

$$E(x) = E_0 \exp(-cx)$$

or

$$\varepsilon(x) = E_0^{-1} \exp(-cs),$$

where E_0, c are positive constants. It is easily verified that

$$\varepsilon'' \geq 0, \quad \varepsilon' < 0.$$

For such a material, the results of Theorem 2 suggest that the spatial decay constant does not exceed that for a homogeneous material.

The interpretation of Theorem 3 in verbal/intuitive terms is also difficult (at least in the generality addressed in the paper). However, we attempt to do so in some particular circumstances. Again, for simplicity, we consider an incompressible material or one with constant Poisson's ratio: consider one which 'hardens' with respect to x for $x < \delta$ (a positive constant) and which 'softens' with respect to x when $x > \delta$, characterized by

$$\varepsilon(x) = \varepsilon_0 \cosh\{c(x - \delta)\},$$

where ε_0, c, δ are positive constants. Then

$$\varepsilon'' \geq 0; \quad \varepsilon' < 0 \quad \text{for} \quad x < \delta, \varepsilon' > 0 \quad \text{for} \quad x > \delta.$$

The results of Theorem 3 suggest that the decay constant arising in these circumstances is less than that arising in the homogeneous case. A simimalar conclusion can be drawn for an incompressible material which 'softens' (uniformly) with respect to x, characterized by

$$\varepsilon(x) = \varepsilon_0 \exp(-cx))$$

where ε_0, c are positive constants.

Appendix

Inequality 1: Any smooth function $\Phi(y)$, $y(0, 1)$, such that $\Phi(0) = \Phi(1) = 0$, satisfies

$$\int_0^1 \varepsilon(y)\Phi'^2 \mathrm{d}y \geq \lambda_1 \int_0^1 \varepsilon\Phi^2 \,\mathrm{d}y\,, \tag{A1}$$

where λ_1 is the lowest eigenvalue of

$$(\varepsilon\Phi^{\cdot})^{\cdot} + \lambda\varepsilon\Phi = 0, \quad \Phi(0) = \Phi(1),$$

or (via the transformation $u = \varepsilon^{1/2}\Phi$) of

$$u^{\cdot\cdot} + [\lambda - (\varepsilon^{1/2})^{\cdot\cdot}\varepsilon^{-1/2}]\,u = 0,\ \ u(0) = u(1) = 0\,.$$

Inequality 2: Any smooth function $\Phi(y)$, $y \in 0, 1$ such that $\Phi(0) = \Phi^{\cdot}(0) = \Phi(1) = \Phi^{\cdot}(1) = 0$ satisfies the inequality

$$\int_0^1 \Phi^{\cdot\cdot 2}\mathrm{d}y \ \geq 4\pi^2 \int_0^1 \Phi^{\cdot\,2}\mathrm{d}y\,. \tag{A2}$$

References

1. J. N. Flavin, Saint-Venant decay estimates for inhomogeneous isotropic two-dimensional elastostatics. In: P.E. O'Donoghue, J.N. Flavin (eds.), *Symposium on Trends in the Application of Mathematics to Mechanics* 2000, Elsevier (2000) pp. 72–79.
2. J. K. Knowles, On Saint-Venant's principle in the two-dimensional linear theory of elasticity. *Arch. Ration. Mech. Anal.* 21 (1966) 1–22.
3. C. O. Horgan and J. K. Knowles, Recent developments concerning Saint-Venant's principle. In: T. Y. Wu and J. W. Hutchinson (eds.), *Advances in Applied Mechanics* Vol. 23. New York: Academic Press (1983) pp. 179–269.
4. C. O. Horgan, Recent developments concerning Saint-Venant's principle : an update, *Appl. Mech. Rev.* 42 (1989) 295–303.
5. C. O. Horgan, Recent developments concerning Saint-Venant's principle: a second update, *Appl. Mech. Rev.* 49 (1996) 101–111.
6. C. O. Horgan, Decay estimates for boundary-value problems in linear and nonlinear continuum mechanics. In: Remigio Russo (ed.), *Mathematical Problems in Elasticity*. World Scientific (1996) pp. 47–89.
7. J. N. Flavin and R. J. Knops, Some convexity considerations for a two-dimensional traction problem, *ZAMP* 17 (1988) 166–176.
8. J. N. Flavin, Qualitative estimates for laminafte-like elastic materials. In: D. F. Parker, A. H. England (eds.), *Proceedings of IUTAM Symposium on Anisotropy, Inhomogeneity, and Nonlinearity in Solid Mechanics*, 1994. Dordrecht: Kluwer Academic Publishers (1995) pp. 339–344.
9. M. J. Pindera, J. Aboudi, A. M. Glaeser, S. M. Arnold (eds), *Use of composites in Multiphased and Functionally Graded Materials*, Part B: Engineering 28 B (1997) 175pp.
10. M. R. Scalpato and C. O. Horgan, Saint Venant decay rates for an isotropic inhomogeneous linearly elastic solid in anti-plane shear. *J. Elasticity* 48 (1997) 145–166.
11. A. M. Chan and C. O. Horgan, End effects in anti-plane shear for an inhomogeneous isotropic linearly elastic semi-infinite strip. *J. Elasticity* 51 (1998) 227–242.
12. C. O. Horgan and R. Quintanilla, Saint Venant end effects in anti-plane shear for functionally graded linearly elastic materials, *Math. Mech. Solids* 6 (2001) 115–132.
13. J. N. Flavin and S. Rionero, *Qualitative Estimates for Partial Differential Equations: An Introduction*. Boca Raton Fl.: CRC Press (1995) 368pp.
14. P. Vafeades and C. O. Horgan. Exponential decay estimates for solutions of the non Kármán equations on a semi-infinite strip. *Arch. Ratl. Mech. Anal*. 104 (1988) 1–25.
15. J. N. Flavin, On Knowles version of Saint Venant's principle in two dimensional elastostatics. *Arch. Ratl. Mech. Anal.* 53 (1994) 366–375.